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Representation of the parametric state of a two-level system on the complex plane

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Abstract. The representation of the transitional state of a two-level system interacting with the radiation field on the complex plane provides a point of comparison between polarization (in the optical sense) and atomic transition processes. This corresponds to the Poincaré representation of polarization in optics. Polarization and dynamics are two aspects of an interaction process. The point on the complex plane obeys a differential equation which is equivalent to the gyroscopic equation of motion for vector $\boldsymbol{r}$. The solution is obtained for the general non-resonant case. It is only the special case of resonance that admits of a simple geometrical representation of z as a point on the complex plane obtained by stereographic projection. This representation brings out the intimate connection between the z representation and the r representation. The dynamical problem can also be represented in terms of the z transformation which corresponds to a unitary transformation.

1. Introduction

The state of a two-level atom interacting with radiation can be represented by a unit vector $\boldsymbol{r}$ introduced by Feynman *et al* (1957). The components of the vector are analogous to the Stokes parameters which are components of a unit vector $\boldsymbol{S}$ specifying the state of polarization in optics. The parametric state of the atom can also be represented by a point on the complex plane in exactly the same way in which optical polarization can be represented on the complex plane (Poincaré representation) (Jauch and Rohrlich 1955). The Schrödinger equation in Hilbert space is represented by a differential equation for the point z on the complex plane and by the gyroscopic equation for $\boldsymbol{r}$ in 'spin space'. There is apparently an asymmetry between optics and atomic dynamics: there seems to be no general equation of motion for $\boldsymbol{S}$. We shall only assert here that Maxwell's field equations for the propagation of an electromagnetic wave through a dielectric medium can be expressed as a gyroscopic equation for the vector $\boldsymbol{S}$. Polarization and dynamics thus appear as two alternative modes of describing an interaction process in optics as well as in atomic dynamics.

2. Quantum dynamical law on the z plane

The wavefunction Ψ is a superposition of the two eigenstates of energy Ψ_a, Ψ_b:

$$\Psi = a\Psi_a + b\Psi_b. \tag{2.1}$$

The eigenstates belong to values of the energy E_a and $E_b (E_a > E_b)$. We assume that the atom has no permanent dipole moments in the eigenstates so that $V_{aa} = V_{bb} = 0$ where V is the interaction potential.

The Schrödinger wave equation leads to the two equations

$$i\hbar\dot{a} = aE_a + bV_{ab}$$
$$i\hbar\dot{b} = bE_b + aV_{ba}.$$

Defining three quantities w_1, w_2, w_3 by

$$\hbar w_1 = V_{ab} + V_{ba}$$
$$\hbar w_2 = i(V_{ab} - V_{ba}) \tag{2.2}$$
$$\hbar w_3 = E_a - E_b = \hbar w_0$$

the equations reduce to the single equation:

$$a\dot{b} - b\dot{a} = iw_0 ab - \tfrac{1}{2}i(w_1 + iw_2)a^2 + \tfrac{1}{2}i(w_1 - iw_2)b^2 \tag{2.3}$$

which leads to

$$\dot{z} = iw_0 z + \tfrac{1}{2}iw_+ - \tfrac{1}{2}iw_- z^2 \tag{2.4}$$

where $z = -b/a$ and $w_\pm = w_1 \pm iw_2$. This equation is equivalent to the Schrödinger equation in Hilbert space and the gyroscopic equation for $\boldsymbol{r}$ in r space introduced by Feynman *et al* (1957). The relation between z and $\boldsymbol{r}$ is given by:

$$z = -\frac{b}{a} = \frac{ba^*}{aa^*} = -\frac{r_1 + ir_2}{1 + r_3} = -\frac{1 - r_3}{r_1 - ir_2}. \tag{2.5}$$

The reciprocal relations are

$$r_1 = -\frac{z + z^*}{1 + zz^*}, \qquad r_2 = \frac{i(z - z^*)}{1 + zz^*}, \qquad r_3 = -\frac{zz^* - 1}{zz^* + 1}. \tag{2.6}$$

The real and imaginary parts of z may be separated out by writing

$$z = (u + iv) \tag{2.7}$$

where

$$u = \frac{-r_1}{1 + r_3}, \qquad v = \frac{-r_2}{1 + r_3}. \tag{2.8}$$

The associated equation of motion of the point z can also be obtained from the gyroscopic equation and the $z - \boldsymbol{r}$ relation (2.5).

3. Solution of the z equation: interaction of the two-level system with radiation

In particular for the interaction of the two-level system with electromagnetic radiation:

$$w_\pm = \frac{\gamma E_0}{2\hbar} e^{\pm iwt} \tag{3.1}$$

for the $\Delta m = \pm 1$ transitions and

$$w_1 = \frac{\mu_{Eab}E}{\hbar}, \qquad w_2 = 0, \qquad w_3 = w_0 \tag{3.2}$$

for the $\Delta m = 0$ transition, where $\gamma = \mu^{+}_{ab}$, $\mu^{+} = (\mu_1 + i\mu_2)$, μ_1, μ_2, μ_3 the dipole moment operators, $\mu_E = \mu_3$, w the radiation frequency and $\frac{1}{2}E_0$ the amplitude of the field (in (3.1)) and E the field in the 3-direction in (3.2). Let us assume the solution of the z equation in the form

$$z = A\,\mathrm{e}^{\mathrm{i}\alpha}.$$

The substitution of this in the z equation gives

$$\alpha = wt \tag{3.3}$$

and leads to the differential equation for the amplitude:

$$\dot{A} + \mathrm{i}A(w - w_0) - \mathrm{i}pA^2 + \mathrm{i}p = 0 \tag{3.4}$$

where

$$p = \begin{cases} \gamma E_0/4\hbar & \text{for } \Delta m = \pm 1 \\ \mu_{Eab}/2\hbar & \text{for } \Delta m = 0. \end{cases}$$

Simple integration gives

$$z = -\frac{b}{a} = \frac{1}{\Omega_E}\{(w - w_0) + \mathrm{i}\Omega\cot[\tfrac{1}{2}(\Omega t + \delta)]\}\,\mathrm{e}^{\mathrm{i}wt} \tag{3.5}$$

where δ is a phase constant and

$$\Omega = [(\Omega_E^2 + (w - w_0)^2]^{1/2} \tag{3.6}$$

$$\Omega_E = (w_1^2 + w_2^2)^{1/2} = \gamma E_0/2\hbar \tag{3.7}$$

for $\Delta m = \pm 1$. From now on we shall write expressions for $\Delta m = \pm 1$ only. Analogous expressions can easily be written out for $\Delta m = 0$. From (2.6):

$$r_1 = -\frac{\Omega_E(w - w_0)}{\Omega^2}[1 - \cos(\Omega t + \delta)]\cos wt + \frac{\Omega_E}{\Omega}\sin(\Omega t + \delta)\sin wt$$

$$r_2 = -\frac{\Omega_E(w - w_0)}{\Omega^2}[1 - \cos(\Omega t + \delta)]\sin wt - \frac{\Omega_E}{\Omega}\sin(\Omega t + \delta)\cos wt$$

$$r_3 = -1 + \frac{\Omega_E^2}{\Omega^2}[1 - \cos(\Omega t + \delta)]. \tag{3.8}$$

3.1. *The case of resonance* $(w - w_0 = 0)$

Initial conditions: we consider the following two initial states specified by the values of r_3 or z at $t = 0$ when the radiation field is applied:

(i) $t = 0, \quad r_3 = -1, \quad z_0 = \infty$ (ground state) (3.9)

(ii) $t = 0, \quad r_3 = +1, \quad z_0 = 0$ (excited state). (3.10)

For the absorptive process ($b \to a$) with the initial condition (i), $\delta = 0$; and for the emissive process ($a \to b$) with the initial condition (ii), $\delta = \pi$. Thus for the absorptive process:

$$z = \mathrm{i} \cot(\tfrac{1}{2}\Omega_E t)\, \mathrm{e}^{\mathrm{i}w_0 t} \tag{3.11}$$

and

$$\begin{aligned} a &= \sin(\tfrac{1}{2}\Omega_E t) \mathrm{e}^{-\frac{1}{2}\mathrm{i}w_0 t} \\ b &= -\mathrm{i} \cos(\tfrac{1}{2}\Omega_E t)\, \mathrm{e}^{\frac{1}{2}\mathrm{i}w_0 t}. \end{aligned} \tag{3.12}$$

(3.12) is correct up to an arbitrary phase constant. For the emissive process:

$$z = -\mathrm{i} \tan(\tfrac{1}{2}\Omega_E t)\, \mathrm{e}^{\mathrm{i}w_0 t} \tag{3.13}$$

$$\begin{aligned} a &= \cos(\tfrac{1}{2}\Omega_E t)\, \mathrm{e}^{\frac{1}{2}\mathrm{i}w_0 t} \\ b &= \mathrm{i} \sin(\tfrac{1}{2}\Omega_E t)\, \mathrm{e}^{\frac{1}{2}\mathrm{i}w_0 t}. \end{aligned} \tag{3.14}$$

The vector components are given by

$$\begin{aligned} r_1 &= \pm \sin(\Omega_E t) \sin(w_o t) \\ r_2 &= \mp \sin(\Omega_E t) \cos(w_0 t) \\ r_3 &= \mp \cos(\Omega_E t), \end{aligned} \tag{3.15}$$

the upper signs correspond to absorption and the lower to emission.

4. Transformations on the complex plane

The transformation of vector $\boldsymbol{r}$ into vector $\boldsymbol{r}'$ is effected by the unitary transformation $\mathbf{T} = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}$ (Venkatesh and Dixit 1971). Expressing the components of $\boldsymbol{r}$ and $\boldsymbol{r}'$ in terms of z and z', we have

$$\frac{1}{z'z'^*+1}\begin{pmatrix} z'z'^*-1 & 2z^{*\prime} \\ 2z' & -(z'z'^*-1) \end{pmatrix} = \frac{1}{zz^*+1}\begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}\begin{pmatrix} zz^*-1 & 2z^* \\ 2z & -(zz^*-1) \end{pmatrix}\begin{pmatrix} a^* & b^* \\ -b & a \end{pmatrix}$$

which leads to the two equations

$$\frac{z'z'^*}{z'z'^*+1} = \frac{1}{zz^*+1}(aa^*zz^* + bb^* - abz^* - a^*b^*z)$$

$$\frac{z'^*}{z'z'^*+1} = \frac{1}{zz^*+1}[ab^*(zz^*-1) + a^2z^* - b^{*2}z].$$

Dividing the first equation by the second:

$$z' = \frac{aa^*zz^* + bb^* - abz^* - a^*b^*z}{ab^*(zz^*-1) + a^2z^* - b^{*2}z}.$$

The right-hand side can be factorized as follows:

$$\frac{a^*z-b}{b^*z+a}\,\frac{az^*-b^*}{az^*-b^*}.$$

Thus the transformation of z is given by

$$z'=\frac{a^*z-b}{b^*z+a}. \tag{4.1}$$

This corresponds to the transformation of $\boldsymbol{r}$ to $\boldsymbol{r}'$ by the unitary matrix $\mathbf{T}$.

Alternatively, using the transformation O_{+3} we obtain directly the law of z transformation (see appendix).

Thus:

$$z'=\frac{r_1'+ir_2'}{1-r_3'}=\frac{2a^*br_3-b^2(r_1-ir_2)+a^{*2}(r_1+ir_2)}{1-(aa^*-bb^*)r_3+ab(r_1-ir_2)+a^*b^*(r_1+ir_2)}$$
$$=\frac{a^*b(zz^*-1)-b^2z^*+a^{*2}z}{bb^*zz^*+aa^*+abz^*+a^*b^*z}$$

which can be factorized as already mentioned to give (4.1).

If the initial state (state of excitation or the ground state) is denoted by z_0, we have

$$z'=\frac{a^*z_0-b}{b^*z_0+a}. \tag{4.2}$$

If we consider the initial condition $z_0\to\infty$, the transformation (4.2) gives $z'=a^*/b^*$, and if the initial condition is $z_0=0$, $z=-b/a$. In either case the process is $a\to b$. The following transformation is obtained from (4.1) by putting $z_0=-1/z_0^*$:

$$z'=\frac{-bz_0^*-a^*}{az_0^*-b^*}. \tag{4.2$'$}$$

This corresponds to the process $b\to a$.

The formula (4.2) is useful in treating problems of interaction of radiation with matter. For instance in the photon echo problem where one considers the response of a two-level system to a sequence of pulses, z_n resulting from n radiation pulses is given by

$$z_n=\frac{T_{22}z_0-T_{21}}{-T_{12}z_0+T_{11}}=\frac{(\mathbf{T}_n\mathbf{T}_{n-1}\ldots\mathbf{T}_2\mathbf{T}_1)_{22}z_0-(\mathbf{T}_n\mathbf{T}_{n-1}\ldots\mathbf{T}_2\mathbf{T}_1)_{21}}{-(\mathbf{T}_n\mathbf{T}_{n-1}\ldots\mathbf{T}_2\mathbf{T}_1)_{12}z_0+(\mathbf{T}_n\mathbf{T}_{n-1}\ldots\mathbf{T}_2\mathbf{T}_1)_{11}} \tag{4.3}$$

where

$$\mathbf{T}=\mathbf{T}_1\mathbf{T}_2\mathbf{T}_3\ldots\mathbf{T}_n$$
$$\mathbf{T}=\begin{pmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{pmatrix}=\begin{pmatrix}a & -b^*\\ b & a^*\end{pmatrix}. \tag{4.4}$$

The proof proceeds by induction; the calculations being omitted. Exactly similar considerations are valid for the determination of the state of polarization of an electromagnetic wave which is propagated through a series of dielectric layers. $\mathbf{T}_\theta$ represents the action of optically active and $\mathbf{T}_\phi$ that of doubly refracting dielectric media (see (5.2)).

5. Geometrical meaning of the *z* transformation

The z transformation (4.1) is the analytic counterpart of the motion of a point z on the complex plane obtained by the stereographic projection outlined in this section (see figure 1).

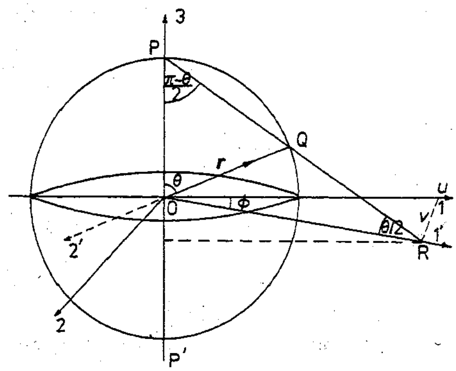

Figure 1.

Also the intimate connection between the z formalism outlined in the foregoing and the $\boldsymbol{r}$ formalism of Feynman *et al* is seen from the geometrical considerations. The expressions (3.15) can be combined into a single set of expressions for the unit vector $\boldsymbol{r}$:

$$
\begin{aligned}
r_1 &= \sin\theta\cos\phi \\
r_2 &= \sin\theta\sin\phi \\
r_3 &= \cos\theta
\end{aligned}
\tag{5.1}
$$

where $\theta = \Omega_E t - \pi$, $\phi = w_0 t + \pi/2$ for the process $b \to a$ and $\theta = \Omega_E t$, $\phi = w_0 t + \pi/2$ for the process $a \to b$. The transformation $\mathbf{T}$ is given by

$$\mathbf{T} = \mathbf{T}(\phi)\mathbf{T}(\theta)\mathbf{T}(\phi)$$

where

$$\mathbf{T}(\theta) = \begin{pmatrix} \cos\frac{1}{2}\theta & \mathrm{i}\sin\frac{1}{2}\theta \\ \mathrm{i}\sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta \end{pmatrix}, \qquad \mathbf{T}(\phi) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\phi/2} \end{pmatrix}. \tag{5.2}$$

Here $\theta = \Omega_E t$ is measured from the positive z axis. $\mathbf{T}$ represents a rotation in r space corresponding to the z transformation (4.2) or (4.2′).

Consider the unit sphere and a unit vector $\boldsymbol{r}$ touching the sphere at Q. The point R(u, v) is the stereographic projection of Q with P as the pole of projection. Here we have three representations of a parametric state—by means of: (i) vector $\boldsymbol{r}$ whose components are r_1, r_2, r_3; (ii) point Q on the unit sphere whose coordinates are r_1, r_2, r_3;

and (iii) point R on the complex plane with the complex coordinate z. They correspond respectively to representations of optical polarization by Stokes vector, or by a point on the Poincaré sphere or by a point on the complex plane. From geometry $\mathrm{OR} = r'$ is given by

$$r' = \cot(\theta/2) \tag{5.3}$$

θ being measured from the line OP. The coordinates of R on the complex plane are

$$\begin{aligned} u &= r' \cos\phi = \cot(\theta/2)\cos\phi \\ v &= r' \sin\phi = \cot(\theta/2)\sin\phi. \end{aligned} \tag{5.4}$$

Thus

$$z = u + \mathrm{i}v = \cot(\tfrac{1}{2}\theta)\,\mathrm{e}^{\mathrm{i}\phi}. \tag{5.5}$$

The gyroscopic equation of motion of $\boldsymbol{r}$ can be written as

$$\begin{aligned} \dot{r}_+ &= \mathrm{i}w_3 r_+ - \mathrm{i}w_+ r_3 \\ \dot{r}_3 &= w_1 r_2 - w_2 r_1 \end{aligned} \tag{5.6}$$

and from the angle representation (5.1)

$$\begin{aligned} w_1 &= -\dot{\theta}\sin\phi \\ w_2 &= \dot{\theta}\cos\phi \\ w_3 &= \dot{\phi}. \end{aligned} \tag{5.7}$$

Differentiating (5.5) and substituting from (5.7) for $\dot{\phi}$ and $\dot{\theta}$ we again obtain the associated equation of motion for z:

$$\dot{z} = \mathrm{i}w_3 z + \tfrac{1}{2}\mathrm{i}w_+ - \tfrac{1}{2}\mathrm{i}w_- z^2.$$

The connection with physics is made by finding the angular velocities from the Schrödinger equation as we have shown in § 2 for z or as shown by Feynman *et al* for $\boldsymbol{r}$.

Comparing (5.5) and (3.11)

$$\begin{aligned} \theta &= \Omega_E t = \frac{\gamma E_0}{2\hbar} t \qquad \text{for } \Delta m = \pm 1 \\ \phi &= wt + \pi/2. \end{aligned} \tag{5.8}$$

6. Summary and discussion

The vectorial method of Feynman *et al* for solving the two-level problem introduces the vector $\boldsymbol{r}$ formed by products of the probability coefficients a, b (and their complex conjugates). The z representation outlined in this paper introduces the *ratio* z, of a to b (and of their complex conjugates) for solving the same problem. The latter method leads to a differential equation for z, equivalent to the Schrödinger equation, and a straightforward analytical solution of the two-level problem. The components of the vector $\boldsymbol{r}$ and the $\mathbf{T}$ matrix are obtained for the non-resonant case and the solutions for resonant absorption (and emission) obtained as special cases.

There is a very close analogy between optical polarization and the dynamics of an atom interacting with radiation. The state of polarization in optics can be represented by a unit vector (Stokes vector) or a point on the unit sphere or a point on the complex plane obtained by stereographic projection (Poincaré representation). Analogously the transitional state of an atom at any instant of time can be represented by a vector and, as has been shown here, by a point Q on the unit sphere at which the vector touches the sphere and by a point on the z plane obtained by a stereographic projection of Q. z is given by $-b/a$ or a^*/b^* obeying the same differential equation. The geometrical picture of the dynamical process obtained in this way brings out clearly the relation between the method of Feynman *et al* and the complex-plane method.

The z transformation (4.2) of the initial state z_0 corresponds to the transformation of $\boldsymbol{r}$ at $t=0$ represented by $\pm\boldsymbol{\sigma}_3 = \pm\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ by the unitary transformation $\left(\begin{smallmatrix}a & -b^*\\ b & a^*\end{smallmatrix}\right)$ and gives the solution of the dynamical problem. The substitution of this transformation in the differential equation for z gives the transformation in terms of the interaction. The transformation is similar to the contact transformation in Lagrangian mechanics and determines the time evolution of the parametric state in accordance with the differential equation. If the unitary transformation $\mathbf{T}_1$ transforms z_0 to z_1 and $\mathbf{T}_2$ transforms z_1 to z_2, then it is easily seen by direct substitution in (4.2) that $\mathbf{T}_1\mathbf{T}_2$ transforms z_0 to z_2 directly. The z method is particularly well adapted to the treatment of the problem of sequential inputs in atomic transition processes as well as the propagation of radiation through a multilayer dielectric.

An assembly of atoms interacting with the thermal field of a crystal lattice and making stepwise transitions in a random manner (Brownian motion) is represented by a cloud of points on the z plane which thus provides a kind of phase space for a statistical assembly of two-level systems.

The main interest of course is the optical analogy. The parallelism between optical polarization and resonant interaction of a two-level system with radiation as regards the basic kinematical and dynamical structure enables one to exploit the formalism of the one to the solution of specific problems in the other (Venkatesh and Roy 1971). In fact the method outlined here provides an important point of comparison between optical polarization and the dynamics of an atom which is in an absorptive or emissive state. But the significance of this parallelism is far-reaching. It points to the quantal nature of the radiation field (Venkatesh and Ram 1976) and leads to the important connection between spin and polarization.

It is emphasized, however, that the analogy with optics is true only for the resonance case of absorption (or emission) under the influence of radiation and it is for this case that the geometrical picture is valid. The solution for $\boldsymbol{r}$ in the non-resonant case has a fairly complicated geometrical representation in r space and perhaps no optical analogue and no simple representation on the complex plane.

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Appendix

The following is the transformation law $\boldsymbol{r} \to \boldsymbol{r}'$:

$$\boldsymbol{r}' = \mathbf{T}\boldsymbol{r}\bar{\mathbf{T}}$$

or

$$\begin{pmatrix} r_3' & r_1' - \mathrm{i}r_2' \\ r_1' + \mathrm{i}r_2' & -r_3' \end{pmatrix} = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \begin{pmatrix} r_3 & r_1 - \mathrm{i}r_2 \\ r_1 + \mathrm{i}r_2 & -r_3 \end{pmatrix} \begin{pmatrix} a^* & b^* \\ -b & a \end{pmatrix}$$

which gives explicitly

$$r_1' = \tfrac{1}{2}[(a^2 + a^{*2}) - (b^2 + b^{*2})]r_1 + \tfrac{1}{2}\mathrm{i}[(a^{*2} - a^2) + (b^2 - b^{*2})]r_2 + (a^*b + ab^*)r_3$$

$$r_2' = \tfrac{1}{2}\mathrm{i}[(a^2 - a^{*2}) + (b^2 - b^{*2})]r_1 + \tfrac{1}{2}[(a^{*2} + a^2) + (b^2 + b^{*2})]r_2 + (a^*b - ab^*)r_3$$

$$r_3' = -(ab + a^*b^*)r_1 + \mathrm{i}(ab - a^*b^*)r_2 + (aa^* - bb^*)r_3.$$

The transformation matrix is O_{+3}.

References

Feynman R P, Vernon F L and Hellworth R W 1957 *J. Appl. Phys.* **28** 49–52
Jauch J M and Rohrlich F 1955 *The Theory of Photons and Electrons* (New York: Addison-Wesley) pp 42–3
Venkatesh H G and Dixit L P 1971 *Ind. J. Phys.* **45** 97–106
Venkatesh H G and Ram J 1976 *J. Phys. A: Math. Gen.* **9** 999–1014
Venkatesh H G and Roy N R 1971 *J. Phys. B: Atom. Molec. Phys.* **4** 408–19